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LIMIT FORM OF THE EQUATION OF ANISOTROPIC HEAT CONDUCTION IN A LAYER

A. I. Moshinskii

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Consideration is given to the problem of asymptotic reduction to a two-dimensional equation of an equation that is three-dimensional along the coordinates and describes the process of heat propagation in an anisotropic material. The region of heat transfer is a layer that is thin along one coordinate. It is assumed that the matrix of the thermal diffusivities depends on the spatial coordinates. The effective thermal-diffusivity matrix is represented in the constructed equivalent heat conduction equation.

Introduction. In the case of heat and mass transfer equations, cases are not infrequent when, because of difficulty of analysis, it is desirable to switch over to a simplified model. This is especially true if averaged (integral) characteristics of the process rather than a detailed field of the temperatures and concentrations in a body are a matter of interest to researchers. In these cases, obtaining simplified equations of the process that are quite exact for practical needs is an attractive feature. It is required of the simplified models that they be similar in a sense to the initial model and enable us to find correction equations when needed.

A good example of this simplification is the Taylor model of effective diffusion (heat conduction) [1, 2], which has gained wide acceptance in describing the transfer of heat and mass in channels, apparatuses, etc. For the average cross-sectional concentration of a substance, Taylor proposed an equation with an effective diffusion (dispersion) coefficient that was calculated from the velocity profile in a channel.The first consideration in transformation of the problem of [1, 2] is the possibility of substantiating it by certain physical and mathematical arguments (computations). This proved to be very attractive for simplification of a mathematical description of heat and mass transfer processes, owing to which the Taylor method was generalized and was improved in different directions (for example, [3-6]). In the problem proposed below (as in the Taylor problem [1-2]), direct averaging of the initial equations does not lead to a desirable result, since terms that cause the averaged problem to be open remain in the equation. Therefore, the main problem will be substantiation of the averaging and obtaining a closed system of equations (equation), error detection, and indication of further steps (if they are necessary) to refine the result.

Formulation of the Problem. Let us analyze the process of propagation of heat in a plane layer of material that is anisotropic as far as the transfer of heat is concerned. More precisely, the region of heat transfer is limited by two planes $Z = 0$ and $Z = H$ and, in general, extends infinitely in the X and Y directions. In fact, we could prescribe a closed cylindrical surface with the equation $S(X, Y) = 0$ that would serve as the boundary of the heat-transfer region together with the noted planes and take a standard boundary condition at this boundary. However this is of no significance for the subsequent presentation. The process of heat propagation in the region is described by the equation

$$\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial X_\mu} \left(a_{\mu\nu} (X, Y, Z) \frac{\partial T}{\partial X_\nu} \right) , \tag{1}$$

where, as is often done in tensor and matrix calculi, summation from unity to three is carried out with respect to the double subscript (in this case, μ and ν). Here for convenience we assumed $X_1 = X$, $X_2 = Y$, and $X_3 = Z$; $a_{\mu\nu}$

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(X, Y, Z) is the thermal-diffusivity tensor, which will be assumed to meet the requirements of nonequilibrium thermo dynamics [7-9], namely, the symmetry $a_{\mu\nu} = a_{\nu\mu}$ (the Onsager relation) and positive definiteness: $a_{\mu\nu}\xi_\mu\xi_\nu \geq \kappa\xi_\mu\xi_\mu$ $(\kappa > 0)$. The latter requirement is associated with the dissipative nature of the process and the production of entropy [7, 8]. As the boundary conditions for Eq. (1) we take the following conditions:

$$a_{z\nu} \frac{\partial T}{\partial X_\nu}\bigg|_{Z=0;H} = 0 , \tag{2}$$

$$T|_{X,Y\to\pm\infty} < \infty . \tag{3}$$

Boundary condition (2) expresses the absence of heat flux in the direction of the external normal to the boundaries $Z = 0$ and $Z = H$ while condition (3) expresses the boundedness of solutions at large distances from the coordinate origin. The extension of the region in the X and Y directions is of no fundamental importance for us, i.e., condition (3) is written for concreteness, to make the formulation of the problem complete. In it, we also adopt the initial condition

$$T|_{\tau=0} = T_n (X, Y, Z) , \tag{4}$$

We note that problems similar to (1)-(4) can arise when mass-transfer problems and the processes of transfer in porous media are investigated [9, 15].

Analysis of the Problem. In this work, simplification of problem (1)-(4) will be the main objective for us. The basic supplementary conditions are (2). In what follows we will indicate complications in the formulation of problem (1)-(4) that do allow for a simplified asymptotic formulation according to the scheme presented.

Let us introduce the operation of averaging of a function F:

$$\langle F (X, Y, \tau)\rangle = \frac{1}{H} \int_0^H F (X, Y, Z, \tau)\, dZ . \tag{5}$$

It should be noted that in the particular case when $a_{\mu z} = 0$ and $a_{zz} = a_{zz}(X, Y, Z)$ while the remaining components of the tensor $a_{\mu\nu}$ are independent of Z, averaging Eq. (1) and conditions (3) and (4) in view of boundary conditions (2) leads directly to a reduction in the dimensionality of the problem, i.e., we obtain a closed problem for the average, according to relationship (5), temperature. Our interest is in a more general, nontrivial case. However we use the specific properties of the region of integration of Eq. (1) when the scales of the corresponding variables differ in magnitude for the region distinctly less extended in the Z direction than in the X and Y directions. More specifically, we write Eq. (1) and boundary condition (2) according to the formulas:

$$x_1 = x = \frac{X}{L_*}, \quad x_2 = y = \frac{Y}{L_*}, \quad x_3 = z = \frac{Z}{H}, \quad A_{\mu\nu} = \frac{a_{\mu\nu}}{a_*}, \quad t = \tau \frac{a_*}{L_*^2}, \quad \frac{H}{L_*}, \tag{6}$$

in dimensionless form

$$\varepsilon^2 \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(A_{zz}\frac{\partial T}{\partial z}\right) + \varepsilon\left[\frac{\partial}{\partial z}\left(A_{iz}\frac{\partial T}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(A_{iz}\frac{\partial T}{\partial z}\right)\right] + \varepsilon^2 \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial T}{\partial x_j}\right), \tag{7}$$

where now and in what follows we will assume that summation from one to two is with respect to the double Latin subscript (with respect to i in the terms for ε and with respect to i and j in the term for ε^2). For example, $A_{ij}\partial T/\partial x_j = \sum_{j=1}^{2} A_{ij}\partial T/\partial x_j$. If the Greek subscript (as μ and ν in (1)) occurs twice summation is from one to three with respect to it. We put emphasis on the subscript z to the z coordinate; we cannot sum with respect to it.

Boundary condition (2) in the variables of (6) acquires the form

$$A_{zz}\frac{\partial T}{\partial z}\bigg|_{z=0;1} + \varepsilon A_{iz}\frac{\partial T}{\partial x_i}\bigg|_{z=0;1} = 0\,. \tag{8}$$

The time scale in (6) is such that the equation of effective heat conduction is constucted with only two spatial coordinates x and y. Therefore, the formula for t is quite natural. Conditions (3) and (4) retain their form in the dimensionless variables of (6) while in the formula for averaging (5) the upper limit in the integral will be unity, and therefore the factor $1/H$ is made unnecessary. By virtue of the above we will not rewrite formulas (3), (4), and (5).

When heat propagates in a layer $\varepsilon \ll 1$, as a rule. Therefore, the perturbation method [11, 12] is a natural method for seeking a solution of the problem for Eq. (7) with supplementary conditions (3), (4), and (8). Thus, we seek a solution of the noted problem in the form of the expansion

$$T = T_0\,(x, y, z, t) + \varepsilon T_1\,(x, y, z, t) + \varepsilon^2 T_2\,(x, y, z, t) + \ldots\,, \tag{9}$$

substituting which into Eq. (7) and boundary condition (8), we obtain the sequence of problems

$$\frac{\partial\,(A_{zz}\partial T_0/\partial z)}{\partial z} = 0\,, \quad A_{zz}\frac{\partial T_0}{\partial z}\bigg|_{z=0;1} = 0\,; \tag{10}$$

$$\frac{\partial}{\partial z}\left(A_{zz}\frac{\partial T_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(A_{iz}\frac{\partial T_0}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}\left(A_{iz}\frac{\partial T_0}{\partial z}\right), \quad \left(A_{zz}\frac{\partial T_1}{\partial z} + A_{iz}\frac{\partial T_0}{\partial x_i}\right)\bigg|_{z=0;1} = 0\,; \tag{11}$$

$$\frac{\partial}{\partial z}\left(A_{zz}\frac{\partial T_k}{\partial z}\right) + \frac{\partial}{\partial z}\left(A_{iz}\frac{\partial T_{k-1}}{\partial x_i}\right) = \frac{\partial T_{k-2}}{\partial t} - \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial T_{k-2}}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(A_{iz}\frac{\partial T_{k-1}}{\partial z}\right), \tag{12}$$

$$\left(A_{ij}\frac{\partial T_k}{\partial z} + A_{iz}\frac{\partial T_{k-1}}{\partial x_i}\right)\bigg|_{z=0;1} = 0\,, \quad k = 2, 3, \ldots$$

after terms of the same order in ε are grouped.

Each equation of (11) and (12) for the functions T_k has a certain necessary condition for the existence of a solution. We denote the sum of all the terms in the right-hand side of Eq. (12) by the symbol F_k. Then, averaging the equation

$$\frac{\partial\,(A_{zz}\,\partial T_k/\partial z)}{\partial z} + \frac{\partial\,(A_{iz}\,\partial T_{k-1}/\partial x_i)}{\partial z} = F_k$$

and taking into account boundary condition (12), we obtain

$$\langle F_k\rangle = 0\,, \quad k = 2, 3, \ldots\,. \tag{13}$$

Integration of Eq. (10) with allowance for the boundary conditions indicates the independence of the variable T_0 on the coordinate z. We denote $T_0 = G(x, y, t)$. In such an event, the right-hand side of Eq. (11) is zero and we can integrate it directly once. Doing this with allowance for boundary condition (11), dividing the result by the positive A_{zz}, and integrating with respect to z once again, we arrive at the following relationship for the function T_1:

$$T_1 = T_1^0\,(x, y, t) - \frac{\partial G}{\partial x_i}\int_0^z dz\,\frac{A_{iz}}{A_{zz}}\,, \tag{14}$$

where $T_1^0(x, y, t)$ is a function of the indicated variables that remains to be defined.

Here we consider only the equation for the principal approximation of expansion (9), for which purpose it will suffice to take $k = 2$ in (13) with allowance for the relationships obtained for the variables T_0 and T_1.

From relations (13) and (14) we have

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial x_i}\left(\langle A_{ij}\rangle \frac{\partial G}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\left\langle \frac{A_{iz}A_{jz}}{A_{zz}}\right\rangle \frac{\partial G}{\partial x_j}\right), \tag{15}$$

where it is taken into account that T_1^0 is independent of z. The form of Eq. (15) yields the formula for the tensor of the effective thermal diffusivities

$$\hat{A}_{ij}(x, y) = \left\langle \frac{A_{ij}A_{zz} - A_{iz}A_{jz}}{A_{zz}} \right\rangle. \tag{16}$$

It is easily established that the elements of the matrix $\hat{A}_{ij}$ are the minors of the matrix $A_{\mu\nu}$ divided by the component A_{zz} (here we allow for the symmetry of the matrix $A_{\mu\nu}$). We note that the principal minors of the matrix $\hat{A}_{ij}$, by virtue of the Sylvester number [13, 14], are positive for the matrix $A_{\mu\nu}$ (including the element A_{zz}); therefore, diagonal elements of the matrix $\hat{A}_{ij}$ will be positive. Since the inverse matrix $A_{\mu\nu}^{-1}$ of a positive-definite matrix is also positive-definite [13, 14], taking into account that its components are the minors of the parent matrix divided by its determinant, which is positive, we can easily see that the determinant of the matrix $\hat{A}_{ij}$ (16) is positive. Thus, the properties of symmetry and positive definiteness are extended to the matrix $\hat{A}_{ij}$ although it has four components instead of nine. This suggests that the matrix $\hat{A}_{ij}$ can qualify well as the thermal-diffusivity matrix.

The asymptotic analysis performed shows that a layer of anisotropic material acts as a new anisotropic (plane) body when the number of space variables is reduced by unity.

Determination of the Initial Condition for Eq. (15). For a complete formulation of the problem, we need to state the initial condition for Eq. (15), which will be written in the final form

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial x_i}\left(\hat{A}_{ij}\frac{\partial G}{\partial x_j}\right). \tag{17}$$

We note that, in fact, we constructed an "external" [11, 12] expansion suitable for description of a process with rather large times. The absence of initial condition (4) in the formulation of the problem for the functions T_j indicates the special (singular) behavior of this expansion. The initial condition dropped out because of the large scale selected in the dimensionless scale $\simeq 1/\varepsilon^2$ in making the time τ dimensionless according to (6). To describe the behavior of a solution with small times, we need to introduce the "contracted" time $\zeta = t/\varepsilon^2$ and to construct a new expansion [11, 12]. The new "internal" [11, 12] problem will be written in the following manner:

$$\frac{\partial T}{\partial \zeta} = \frac{\partial}{\partial z}\left(A_{zz}\frac{\partial T}{\partial z}\right) + \varepsilon\left[\frac{\partial}{\partial z}\left(A_{iz}\frac{\partial T}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(A_{iz}\frac{\partial T}{\partial z}\right)\right] + \varepsilon^2\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial T}{\partial x_j}\right),$$

$$A_{zz}\frac{\partial T}{\partial z}\bigg|_{z=0;1} + \varepsilon A_{iz}\frac{\partial T}{\partial x_i}\bigg|_{z=0;1} = 0, \quad T|_{\zeta=0} = T_n(x, y, z). \tag{18}$$

As before we restrict ourselves to the principal approximation of the internal expansion

$$T = \overline{T}_0(x, y, z, \zeta) + \varepsilon\overline{T}_1(x, y, z, \zeta) + \dots, \tag{19}$$

where the over bar denotes an internal solution. An equation for the function $\overline{T}_0$ of the principal approximation is obtained by the simple substitution of $\varepsilon = 0$ into (18). Thus, we have the problem

$$\frac{\partial \overline{T}_0}{\partial \zeta} = \frac{\partial}{\partial z}\left(A_{zz}\frac{\partial \overline{T}_0}{\partial z}\right), \quad A_{zz}\frac{\partial \overline{T}_0}{\partial z}\bigg|_{z=0;1} = 0, \quad \overline{T}_0|_{\zeta=0} = T_n(x, y, z). \tag{20}$$

For our purposes (joining with the solution of external problem (9)), it will suffice to determine only the average value of the function $\overline{T}_0$: $\langle\overline{T}_0\rangle$. To do this, we average Eq. (20). As before we obtain a zero average value of the operator in the right-hand side. Hence we find

$$\left\langle \frac{\partial \overline{T}_0}{\partial \zeta} \right\rangle = 0 \to \langle \overline{T}_0 \rangle = \text{const}\,(\zeta) = \langle T_n \rangle\,, \tag{21}$$

here we resorted to initial condition (20). Now, employing (21), we use the principle of limit joining [11, 12] for the terms average of the principal terms of the external and internal expansions

$$\lim_{t\to 0} \langle T_0 \rangle = \lim_{t\to 0} G(x, y, t) = G|_{t=0} = \lim_{\zeta\to\infty} \langle \overline{T}_0 \rangle = \langle T_n \rangle = G_n(x, y)\,. \tag{22}$$

Thus, the function $T_0(x, y, t)$ at the initial instant is equal to the average value of the initial function of the initial problem. This is to be expected from intuitive considerations. So, the asymptotic problem of the principal approximation of the external solution reduces to Eq. (17), boundary conditions for the variables X and Z (3) (it is easy to perform a similar analysis for boundary conditions that are different from (3)), and the initial condition

$$G|_{t=0} = \langle T_n \rangle = G_n(x, y)\,. \tag{23}$$

One would expect that the external expansion is of prime interest for practice because of the larger characteristic time of a variation in the parameters. The value of the internal solution reduces to the construction on its basis of the expansions required for joining with the solution of the external problem. Using the joining procedure in the principal approximation of the external solution we were able to obtain a closed problem (independent of the characteristics of the internal solution). We can attempt to obtain the same result (if we are fortunate) in subsequent approximations. We note, however, the following interesting aspect of the internal problem. It describes a rapid ($t = O(\varepsilon^2)$) process of equalization of the concentration of a substance along the z coordinate transverse to the layer. Thus, the dimensionless homogenation time (equalization time of the characteristics of the problem along the z coordinate) in the system is on the order of ε^2.

Final Comments. The proposed transformation of the equation of anisotropic heat conduction is generalized in a natural manner to the presence of heat sources inside the layer and heat flux through its boundaries $z = 0$ and $z = 1$. It is particularly simple to allow for the heat sources when the size scales of these sources are in agreement the characteristic time scale of the problem. More precisely, in the dimensionless variables of (6), the terms that describe the heat sources are on the order of ε^2. In this case, the term $\varepsilon^2 Q(T, x, y, z)$, which represents the volumetric heat source, will enter the right-hand side of Eq. (7) while boundary conditions (8) will take the form

$$\left(A_{zz} \frac{\partial T}{\partial z} + \varepsilon A_{iz} \frac{\partial T}{\partial x_i} \right) \Bigg|_{z=0} = -\,\varepsilon^2 q_0(T, x, y)\,,$$

$$\left(A_{zz} \frac{\partial T}{\partial z} + \varepsilon A_{iz} \frac{\partial T}{\partial x_i} \right) \Bigg|_{z=1} = -\,\varepsilon^2 q_1(T, x, y)\,,$$

where the functions $q_0(T, x, y)$ and $q_1(T, x, y)$ describe the heat fluxes at the boundaries of the region. It is easy to verify that, by following the algorithm described, we will arrive at the following equation of effective thermal conductivity:

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial x_i} \left(\hat{A}_{ij} \frac{\partial G}{\partial x_j} \right) + \langle Q(G, x, y) \rangle + q_0(G, x, y) + q_1(G, x, y)$$

with the same matrix of effective thermal diffusivity that was found before. It is clear that the averaged heat source Q will depend only on the functions G and the coordinates x and y. It is precisely these arguments that remain in the function of the heat source in the given equation.

Of definite interest is the representation of asymptotically averaged equations in some other (non-Cartesian) orthogonal coordinates. Here we give only the corresponding formulas for a layer of $Z = 0$; $Z = H$ in the cylindrical coordinates Z, R, and φ. The basic equation that replaces (7) will be written in the form

$$\varepsilon^2 \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(A_{zz}\frac{\partial T}{\partial z}\right) + \varepsilon\left[\frac{\partial}{\partial z}\left(A_{rz}\frac{\partial T}{\partial r} + \frac{A_{\varphi z}}{r}\frac{\partial T}{\partial \varphi}\right) + \frac{\partial}{r\partial r}\left(rA_{rz}\frac{\partial T}{\partial z}\right) + \right.$$

$$\left. + \frac{\partial}{r\partial\varphi}\left(A_{\varphi z}\frac{\partial T}{\partial r}\right)\right] + \varepsilon^2\left[\frac{\partial}{r\partial r} r\left(A_{rr}\frac{\partial T}{\partial r} + \frac{A_{r\varphi}}{r}\frac{\partial T}{\partial\varphi}\right) + \frac{\partial}{r\partial\varphi}\left(A_{r\varphi}\frac{\partial T}{\partial r} + \frac{A_{\varphi\varphi}}{r}\frac{\partial T}{\partial\varphi}\right)\right]. \quad (24)$$

Instead of boundary conditions (8) we have the following conditions:

$$A_{zz}\frac{\partial T}{\partial z}\bigg|_{z=0;1} + \varepsilon\left[A_{rz}\frac{\partial T}{\partial r} + \frac{A_{\varphi z}}{r}\frac{\partial T}{\partial\varphi}\bigg|\right]_{z=0;1} = 0. \quad (25)$$

Here the dimensionless coordinate r is related to R in the same manner as x and y are expressed earlier in terms of X and Y (6), i.e., $r = R/L$.

Following the scheme presented earlier, we will bring Eq. (24), in view of (25), to the form

$$\frac{\partial G}{\partial t} = \frac{\partial}{r\partial r} r\left(\hat{A}_{rr}\frac{\partial G}{\partial r} + \frac{\hat{A}_{r\varphi}}{r}\frac{\partial G}{\partial\varphi}\right) + \frac{\partial}{r\partial\varphi}\left(\hat{A}_{r\varphi}\frac{\partial G}{\partial r} + \frac{\hat{A}_{\varphi\varphi}}{r}\frac{\partial G}{\partial\varphi}\right), \quad (26)$$

where $\hat{A}_{rr} = \langle A_{rr}\rangle - \langle A_{rz}^2/A_{zz}\rangle$; $\hat{A}_{r\varphi} = \langle A_{r\varphi}\rangle - \langle A_{\varphi z}A_{rz}/A_{zz}\rangle$; $\hat{A}_{\varphi\varphi} = \langle A_{\varphi\varphi}\rangle - \langle A_{\varphi z}^2/A_{zz}\rangle$. As in deriving Eq. (15), use was made of some properties of the matrix $A_{\mu\nu}$, more precisely, its positive definiteness and the conditions of symmetry $A_{rz} = A_{zr}$, etc. It can easily be seen that the initial condition for Eq. (26) will be (23), where the arguments x and y in the function G_n should be replaced by r and φ.

NOTATION

$a_{\mu\nu}$ and $A_{\mu\nu}$, dimensional and dimensionless thermal-diffusivity tensors, respectively ($\mu, \nu = 1, 2, 3$); a_*, scale of thermal-diffusivity tensor; $\hat{A}_{ij}$, effective thermal-diffusivity tensor ($i, j = 1, 2$); G, first term of the expansion of the temperature in terms of ε; H, layer thickness; L_*, scale for the variables X, Y, and R; T, temperature; T_j, terms of the expansion of the temperature in terms of ε (9); $\overline{T}_j$, components of the internal expansion of the temperature in terms of the perturbation ε (19); t, dimensionless time; X_μ, Cartesian coordinates ($\mu = 1, 2, 3$); $\varepsilon = H/L_*$, perturbation; $\zeta = t/\varepsilon^2$, internal time; τ, dimensional time; $\langle\ \rangle$, averaging sign.

REFERENCES

1. G. Taylor, *Proc. Roy. Soc.*, Ser. A, **219**, No. 1137, 186-206, London (1953).
2. G. Taylor, *Proc. Roy. Soc.*, Ser. A, **223**, No. 1155, 446-458, London (1954).
3. R. Aris, *Proc. Roy. Soc.*, Ser. A, **235**, No. 1200, 67-77, London (1956).
4. V. I. Maron, *Zh. Prikl. Mekh. Tekh. Fiz.*, No. 5, 96-102 (1971).
5. V. V. Dil'man and A. E. Kronberg, *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 1, 81-86 (1984).
6. A. I. Moshinskii, *Inzh.-Fiz. Zh.*, **56**, No. 6, 931-936 (1989).
7. S. De Groot and P. Mazur, *Nonequilibrium Thermodynamics* [Russian translation], Moscow (1964).
8. I. Diarmati, *Nonequilibrium Thermodynamics. The Field Theory and Variational Principles* [Russian translation], Moscow (1974).
9. A. V. Luikov, *Heat and Mass Transfer (handbook)* [in Russian], Moscow (1978).
10. E. S. Romm, *Structural Models of the Pore Space of Rocks* [in Russian], Leningrad (1985).
11. J. Cole, *Perturbation Methods in Applied Mathematics* [Russian translation], Moscow (1972).
12. A. Naife, *Perturbation Methods* [Russian translation], Moscow (1976).
13. I. M. Gel'fand, *Lectures on Linear Algebra* [in Russian], 3rd edition, Moscow (1966).
14. R. Bellman, *Introduction to Matrix Theory* [Russian translation], Moscow (1976).